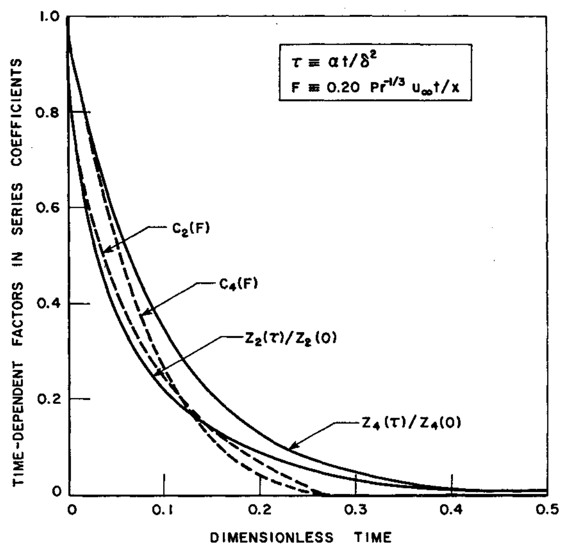

Fig. 1 Variations of the factors $z_2(\tau)/z_2(0)$, $z_4(\tau)/z_4(0)$, $C_2(F)$, and $C_4(F)$ with dimensionless times τ and F

the factors $C_2(F)$ and $C_4(F)$, obtained for boundary layer flows using an integral method, are seen to differ only slightly from values of the factors $z_2(\tau)/z_2(0)$ and $z_4(\tau)/z_4(0)$ obtained for Couette flows using an exact method. Note that, if the time elapsed since quasi-steady conditions existed is small, i.e., if $F \ll 1$ and $\tau \ll 1$, then the terms corresponding to deviations from the quasi-steady heat transfer rate in Eqs. (7) and (10) disappear, whereas the corresponding terms in Eq. (8) do not. This situation is in keeping with the fact that Refs. 1 and 3 take into account the past history of the system, whereas Ref. 2 does not.

An expression more useful than Eqs. (7, 8, and 10) would be obtained if one could combine the most useful features of these equations into a single expression. One might combine those features of Eq. (7) which are unique to an exact treatment of the past history of the system, those features of Eq. (8) which are unique to an exact treatment of the two-dimensional nature of compressible boundary layer flows, and those features of Eq. (10) which are unique to a treatment in which the value of the Prandtl number is arbitrary, and write

$$q_{\text{inst}} = q_{\text{quasi-steady}}\left[1 + 2.67\ Pr^{1/3}\frac{f'(t)}{f(t)}\left(\frac{x}{u_\infty}\right) - 1.00\ Pr^{2/3}\frac{f''(t)}{f(t)}\left(\frac{x}{u_\infty}\right)^2 - 2.67\ \frac{z_2(F)}{z_2(0)}\ Pr^{1/3}\frac{f'(0)}{f(t)}\left(\frac{x}{u_\infty}\right) + 1.00\ \frac{z_4(F)}{z_4(0)}\ Pr^{2/3}\frac{f''(0)}{f(t)}\left(\frac{x}{u_\infty}\right)^2 + \ldots\right] \quad (11)$$

If $Pr^{1/3}\, x/u_\infty$ is replaced by $\delta^2/8\alpha$, then Eq. (7) is retrieved to fair approximation; if $F \gg 1$ and $Pr = 0.72$, then Eq. (8) is retrieved exactly, whereas, if $z_2(F)/z_2(0)$ and $z_4(F)/z_4(0)$ are replaced by $C_2(F)$ and $C_4(F)$, respectively, then Eq. (10) is retrieved to fair approximation.

Fortunately, and as indicated in Refs. 1 and 2, the aerothermodynamicist will find that most forced-convection heat transfer rates to surfaces with time-dependent temperatures may be calculated to good approximation using equations developed for the steady-state case.

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Orbital Transfer in Minimum Time

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THE problem of orbital transfer discussed here is that of scheduling the direction p of constant momentum thrust of a rocket, which loses mass at a constant rate, so that it transfers to a known earth satellite orbit in a minimum time T after launching. The launching conditions are assumed to be fixed. Figure 1 illustrates the problem for a circular orbit. To aid the discussion, imaginary physical rendezvous of the rocket and a target satellite is assumed to occur at the transfer sector angle B. The calculus of variations problem is set up and solved numerically by the method of Faulkner.[1] The nonrotating Oxy axes shown in Fig. 1 are used. The coordinates and velocity components of the rocket and target satellite are denoted by x,y,u,v and X,Y,U,V, respectively. For simplicity the equations of motion of rocket and target will be written in a nondimensional form by using suitable units. The unit of length is taken as the earth's equatorial radius, R_e = 20,925,000 ft. The unit taken for time t is the time required by a hypothetical earth satellite, in equatorial, circular, vacuum, sea-level orbit, to traverse a sector of 1 rad. This unit of time is $(R_e/g)^{1/2}$ = 13.459 min, where g = 32.086 ft/sec² is the acceleration of gravity at the equator. The unit of velocity is then the speed of this hypothetical satellite. These units of length and time always will be understood, unless other units are mentioned specifically.

Statement of the Problem

The equations of motion of the rocket are

$$\begin{aligned} \varphi_1 &= \dot{u} - g_1 - a\cos p = 0 \qquad & \varphi_3 &= \dot{x} - u = 0 \\ \varphi_2 &= \dot{v} - g_2 - a\sin p = 0 \qquad & \varphi_4 &= \dot{y} - v = 0 \end{aligned} \quad (1)$$

where $g_1 = -x/r^3$, $g_2 = -y/r^3$, $r^2 = x^2 + y^2$, and $a = c\dot{M}/(1 - \dot{M}t)g$, where $\dot{M}$ is the constant fraction of initial gross rocket mass lost per unit time, and c is the constant speed of the emitted rocket gases. The fixed launching conditions are taken as

$$\begin{aligned} x(0) &= 0 \qquad & u(0) &= V_1 = V_0\cos\theta \\ y(0) &= 1 \qquad & v(0) &= V_2 = V_0\sin\theta \end{aligned} \quad (2)$$

The terminal point of the rocket trajectory is variable with

$$\begin{aligned} x(T) &= X(T) \qquad & y(T) &= Y(T) \\ u(T) &= U(T) \qquad & v(T) &= V(T) \end{aligned} \quad (3)$$

Note that Eqs. (1) and (3) imply that $\dot{U}(T) = -[X/R^3]_T = g_1(T)$ and $\dot{V}(T) = -[Y/R^3]_T = g_2(T)$, where $R^2 = X^2 + Y^2$. It is assumed also that the rocket thrust is turned off abruptly at time T. The problem is to choose the control variable p to

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effect orbital transfer with T minimized and to determine the corresponding trajectory. This problem is equivalent to the Lagrange calculus of variations problem of requiring the integral

$$I = \int_0^T (1 + \lambda\varphi_1 + \mu\varphi_2 + \pi\varphi_3 + \rho\varphi_4)dt \tag{4}$$

to be stationary. As explained in Ref. 2, Eqs. (1) are regarded as constraints with $\lambda(t)$, $\mu(t)$, $\pi(t)$, $\rho(t)$ introduced in Eq. (4) as continua of Lagrangian multipliers. If time at the varied terminal point is taken as $T + \Delta T$, the vanishing variation of I according to Ref. 3 is

$$\delta I = \int_0^T [\lambda(\delta\dot{u} - g_{1x}\delta x - g_{1y}\delta y + a \sin p\, \delta p) + \pi(\delta\dot{x} - \delta u) + \mu(\delta\dot{v} - g_{2x}\delta x - g_{2y}\delta y - a\cos p\, \delta p) + \rho(\delta\dot{y} - \delta v)]\, dt + \int_T^{T+\Delta T} [1 + \lambda\delta\dot{u} + \mu\delta\dot{v} - (\lambda\cos p + \mu\sin p)\,\delta a]dt = 0 \tag{5}$$

where the $\delta\dot{u}$, $\delta\dot{v}$, and $\delta a = a$ finite variation terms in the integral from T to $T + \Delta T$, created by thrust termination at T on the unvaried trajectory and at $T + \Delta T$ on the varied trajectory, cancel. On integrating by parts, one obtains

$$\delta I = [\lambda\delta u + \mu\delta v + \pi\delta x + \rho\delta y]_T - \int_0^T [(\dot{\lambda} + \pi)\delta u + (\dot{\mu} + \rho)\delta v + (\dot{\pi} + g_{1x}\lambda + g_{2x}\mu)\delta x + (\dot{\rho} + g_{1y}\lambda + g_{2y}\mu)\delta y + a\,(\mu\cos p - \lambda\sin p)\delta p]dt + \Delta T = 0 \tag{6}$$

The variations of the dependent coordinates at the variable terminal point must be taken, according to Ref. 3, as

$$\begin{aligned} \delta u(T) &= (\dot{U} - \dot{u})_T\Delta T = -(a\cos p)_T\Delta T \\ \delta x(T) &= (\dot{X} - \dot{x})_T\Delta T = 0 \\ \delta v(T) &= (\dot{V} - \dot{v})_T\Delta T = -(a\sin p)_T\Delta T \\ \delta y(T) &= (\dot{Y} - \dot{y})_T\Delta T = 0 \end{aligned} \tag{7}$$

Substitution of Eqs. (7) into Eq. (6) yields the Euler equations

$$\begin{array}{lll} \dot{\pi} + g_{1x}\lambda + g_{2x}\mu = 0 & \dot{\rho} + g_{1y}\lambda + g_{2y}\mu = 0 & \\ \dot{\lambda} + \pi = 0 & \dot{\mu} + \rho = 0 & \tan p = \mu/\lambda \end{array} \tag{8}$$

and the transversality condition

$$[a(\lambda\cos p + \mu\sin p)]_T = 1 \tag{9}$$

According to Ref. 4, the first four homogeneous equations of the Euler Eqs. (8) constitute the system adjoint to the variations of Eqs. (1), which are the coefficients of λ, μ, π, ρ in the first integral of Eq. (5) equated to zero. The last of the Euler Eqs. (8) requires that the control variable p be adjusted so that the vector $\mathbf{a} = a\,(\mathbf{i}\cos p + \mathbf{j}\sin p)$, which is proportional to the rocket thrust, is continually parallel to the adjoint vector $\boldsymbol{\Lambda} = \mathbf{i}\lambda + \mathbf{j}\mu$. The transversality Eq. (9), expressible as the dot product $(\mathbf{a}\cdot\boldsymbol{\Lambda})_T = 1 > 0$, requires that $\mathbf{a}$ and $\boldsymbol{\Lambda}$ have the same sense. Since it is only the ratio of μ to λ which determines p, and since they satisfy homogeneous Eqs. (8), it is trivial to multiply each by the same constant to satisfy the magnitude requirement of Eq. (9). A solution of the problem obtained from Eqs. (1) and (8) guarantees a stationary time of transfer. This is a minimum time from the nature of the problem.

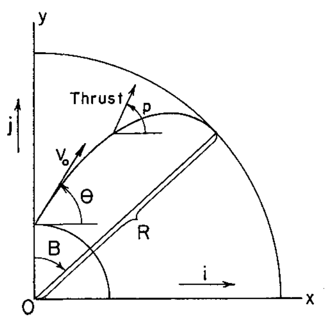

Fig. 1 Orbital transfer

Numerical Solution

There is a constructive aspect of a modification of Eq. (6), first used by Bliss[5] for differential corrections in ballistics and applied recently by Faulkner[1] in an iterative fashion in optimum control problems. To find the desired modification, assume that a solution of Eqs. (1) and (8) has been obtained, which does not necessarily satisfy the terminal Eqs. (3). Using this solution and holding T fixed, find the variation of the vanishing integral

$$\int_0^T (\lambda\varphi_1 + \mu\varphi_2 + \pi\varphi_3 + \rho\varphi_4)dt = 0 \tag{10}$$

with the terminal constraints of Eqs. (3) removed. Since λ, μ, π, ρ satisfy the adjoint system, one obtains

$$[\lambda\delta u + \mu\delta v + \pi\delta x + \rho\delta y]_T = \int_0^T a(-\lambda\sin p + \mu\cos p)\delta p dt \tag{11}$$

Equation (11), which is the desired modification of Eq. (6), is called the fundamental formula by Bliss.[5] By its use it is possible to generate the control parameters of a varied trajectory that, hopefully, comes closer to satisfying the desired terminal Eqs. (3). To do this, assume that the adjoint system has been solved for a set of four linearly independent solutions given by the rows of the 4×4 matrix

$$\text{transpose of } \mathbf{E}(t) = [\lambda_i, \mu_i, \pi_i, \rho_i] \qquad i = 1,2,3,4 \tag{12}$$

where $\mathbf{E}(0) = I$ is the identity matrix. The components of the adjoint vector $\boldsymbol{\Lambda}$, required to satisfy the last of the Euler Eqs. (8), are taken as the linear combinations $\lambda = \lambda_1 + l\lambda_2 + m\lambda_3 + n\lambda_4$ and $\mu = \mu_1 + l\mu_2 + m\mu_3 + n\mu_4$, so that the control angle p is determined by

$$\tan p = (\mu_1 + l\mu_2 + m\mu_3 + n\mu_4)/(\lambda_1 + l\lambda_2 + m\lambda_3 + n\lambda_4) \tag{13}$$

and its variation by

$$\delta p = [(\lambda\mu_2 - \mu\lambda_2)\delta l + (\lambda\mu_3 - \mu\lambda_3)\delta m + (\lambda\mu_4 - \mu\lambda_4)\delta n]/(\lambda^2 + \mu^2) \tag{14}$$

When Eqs. (14) and the rows of matrix (12) are substituted into Eq. (11), there results the system of Bliss formulas

$$[\delta u, \delta v, \delta x, \delta y]_T\, \mathbf{E}(T) = [0, \delta l, \delta m, \delta n]\mathbf{A} \tag{15}$$

where the elements of the matrix $\mathbf{A}$ are

$$a_{ij} = \int_0^T a(\lambda\mu_i - \mu\lambda_i)(\lambda\mu_j - \mu\lambda_j)(\lambda^2 + \mu^2)^{-3/2}dt \tag{16}$$

The coordinates of the varied trajectory terminal point at time $T + \Delta T$ may be taken as $[u + \Delta u, v + \Delta v, x + \Delta x, y + \Delta y]_T$, where

$$[\Delta u, \Delta v, \Delta x, \Delta y]_T = [\delta u, \delta v, \delta x, \delta y]_T + (\dot{u}, \dot{v}, \dot{x}, \dot{y})_T\Delta T \tag{17}$$

In an effort to make this new terminal point come closer to satisfying the terminal Eqs. (3), one may take

$$[\Delta u, \Delta v, \Delta x, \Delta y]_T = [U - u, V - v, X - x, Y - y]_T + [\dot{U}, \dot{V}, \dot{X}, \dot{Y}]_T\Delta T \tag{18}$$

Substitute Eqs. (17) and (18) into Eq. (15) to obtain

$$-[\dot{U} - \dot{u}, \dot{V} - \dot{v}, \dot{X} - \dot{x}, \dot{Y} - \dot{y}]_T\Delta T + [0, \delta l, \delta m, \delta n]\mathbf{A}\mathbf{E}^{-1}(T) = [U - u, V - v, X - x, Y - y]_T \tag{19}$$

as the Newton equations for ΔT, δl, δm, δn on the varied trajectory. The Faulkner[1] scheme for solving optimum control problems may now be stated: make initial estimates for T, l, m, n, and B; make as imultaneous numerical integration of Eqs. (1, 8, and 16) using the Eq. (13) control variable; solve Eq. (19) for ΔT and the control parameter changes δl, δm, δn; iterate until convergence is obtained.

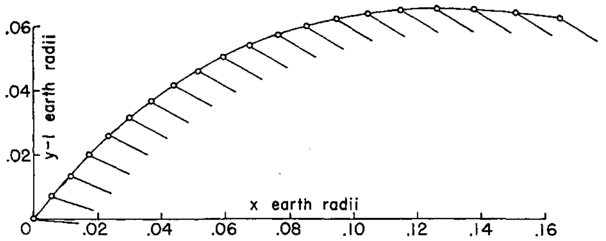

Fig. 2 Trajectory and thrust directions

To give an example, assume a circular orbit of radius $R = 1.075699$ earth radii, corresponding to an altitude of 300 miles. For this orbit $U^2 + V^2 = R^{-1}$ and $\dot{U}^2 + \dot{V}^2 = R^{-4}$. Assume launching conditions $V_0 = 0.585$ and $\theta = 0.928$ rad. Also assume $c = 10{,}000$ fps and $\dot{M} = 0.0036$ sec^{-1}. Use initial estimates $T = 0.29$, $l = -0.22$, $m = -30$, $n = 19$, and $B = 0.154$ rad. This input gave convergence in five iterations to the six-figure results $T = 0.289869$, $l = -0.0751076$, $m = -51.8646$, $n = 32.9787$, and $B = 0.153882$. Figure 2 shows the trajectory and thrust directions for equal time intervals. An approximate body of knowledge about possible combinations of V_1, V_2, $\dot{M}$, T, l, m, n, and B must be built up in order to choose reasonable estimated input to the iteration of Eq. (19). To do this, solve Eqs. (1, 2, and 8) with g_1 and g_2 linearized to obtain

$$x = \int_0^t a \cos p \sin(t - w)dw + V_1 \sin t$$

$$2^{1/2}y = \int_0^t a \sin p \sinh 2^{1/2}(t - w)dw + V_2 \sinh 2^{1/2}t - 2^{-1/2} \cosh 2^{1/2}t + 3/2^{1/2} \tag{20}$$

$$\tan p = (l \cosh 2^{1/2}t - 2^{-1/2}n \sinh 2^{1/2}t)/(\cos t - m \sin t)$$

Using Eqs. (20) and (3), it is easy to solve the Newton equations:

$$(X - x)_T = \Delta V_1 \sin T - Y(T)dB + \Delta c \int_0^T a \cos p \sin(T - w) \frac{dw}{c}$$

$$(U - u)_T = \Delta V_1 \cos T - V(T)dB + \Delta c \int_0^T a \cos p \cos(T - w) \frac{dw}{c}$$

$$2^{1/2}(Y - y)_T = \Delta V_2 \sinh 2^{1/2}T + 2^{1/2}X(T)dB + \Delta c \int_0^T a \sin p \sinh 2^{1/2}(T - w) \frac{dw}{c} \tag{21}$$

$$(V - v)_T = \Delta V_2 \cosh 2^{1/2}T + U(T)\, dB + \Delta c \int_0^T a \sin p \cosh 2^{1/2}(T - w) \frac{dw}{c}$$

$$10{,}000 \ln[1 - (\dot{M} + \Delta\dot{M})T] = (c + \Delta c) \ln(1 - \dot{M}T)$$

for V_1, V_2, $\dot{M}$, and B for assumed sets of T, l, m, and n. The last of Eqs. (21), which swaps ΔM for Δc by holding the integral of a invariant, was appended to simplify computations.

The methods presented here can be used also when the additional constraint of fixing the angle B is imposed.

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Comments

Comment on "Roll Damping of a Fleet Ballistic-Missile Submarine"

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A METHOD has been proposed[1] for estimating the roll-damping moment due to the sail of a fleet ballistic-missile submarine. In the derivation of this method, two-dimensional airfoil theory is used, although the submarine sail is essentially a low-aspect-ratio hydrofoil. As a result, it is believed that the method of Ref. 1 gives results that are greatly in error.

The effect of aspect ratio on roll-damping coefficient (illustrated in Fig. 1) is based on experimental data[2] for untapered, unswept wings. There the coefficient was defined as

$$C_{l_p} = \frac{\partial C_l}{\partial(pb/2V)}$$

since the damping force was found to be linear with roll rate. C_l is the roll-moment coefficient, b the span, p the rolling angular velocity in radians per second, and V the forward velocity. If it is assumed that the submarine-hull diameter is included in the calculation of aspect ratio, the aspect ratio of the submarine used as a numerical example in Ref. 1 is 1.41. From Fig. 1 it is seen that the corresponding C_{l_p} is about -0.09, whereas for the largest value of aspect ratio tested it is -0.37. Thus, assuming steady roll rate, the roll-damping moment in the example is too large by a factor of 4, at least.

Of course, because of the oscillatory motion of the submarine, the full steady-state lift is not developed. However, for very low aspect ratios the lift appears very rapidly when the hydrofoil is given a sudden change in angle of attack. This is illustrated[3] by Fig. 2, which represents the fraction of steady-state lift attained as a function of the distance in chords which the hydrofoil has moved after a sudden change in angle of attack. Reference 3 indicates that, in the limiting case, aspect ratio zero, the steady-state lift is attained immediately. Interpolating, for aspect ratio 1.41, over 75% of the steady lift appears immediately. Since the roll-damping

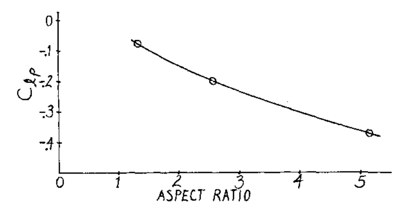

Fig. 1 Variation of roll-damping coefficient with aspect ratio

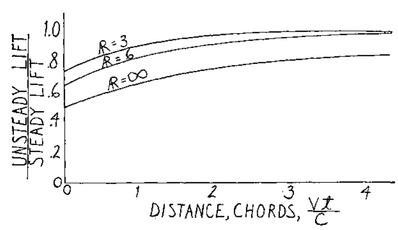

Fig. 2 Unsteady-lift function

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